

# Conversion of entanglement between continuous variable and qubit systems

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## Abstract

We investigate how entanglement can be transferred between continuous variable and qubit systems. We find that a two-mode squeezed vacuum state and a continuous variable Werner state can be converted to the product states of infinitive number of two-qubit states while keeping the entanglement. The reverse process is also possible.

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## 1 Introduction

Quantum information processing (QIP) has been extensively studied for a qubit system which is a quantum extension of a bit, spanning two-dimensional Hilbert space. A qubit is realized by a electronic spin, a two-level atom, the polarization of a photon and a superconductor among others. Parallelly, much attentions have been paid to the QIP of quantum continuous variable (CV) system which is a quantum extension of analog information in classical information theory. CV physical systems such as a harmonic oscillator, a rotator and a light field are defined in infinitive-dimensional Hilbert space. While conversions of analog to digital (A/D) and digital to analog (D/A) are quite usual in information processing, qubit and CV systems are nearly always treated separately. There have been some pilot works on how to entangle two separate qubits by an entangled Gaussian field, the efficient of the transfer is not high [1]. We would propose a scheme of perfect transferring the entanglement in this paper.

## 2 Entanglement conversion of two mode squeezed vacuum state

The two-mode squeezed vacuum state  $|\Psi\rangle_{AB} = \sqrt{1-\lambda^2} \sum_{m=0}^{\infty} \lambda^m |m\rangle_A \otimes |m\rangle_B$ , where  $\lambda = \tanh r$  with  $r$  the squeezing parameter. The entanglement of the state is  $E(|\Psi\rangle) = \cosh^2 r \log \cosh^2 r - \sinh^2 r \log \sinh^2 r$ . The interaction between different systems can cause the transfer of entanglement between the systems. The scheme of the system considered is that two individual qubits each interacting with one entangled part of the field. The whole system will evolve in the way of  $U(t) \rho_{AB}(0) \otimes \rho_{CD}(0) U^+(t)$ , where  $U(t) = \exp[-\frac{i}{\hbar}(H_{AC}+H_{BD})t]$  is the evolution operator in interaction picture, and  $\rho_{AB}(0) = |\Psi\rangle_{AB} \langle \Psi|$  is the initial state of the CV system while  $\rho_{CD}(0) = \prod_{k=1}^K |-\rangle_{CD}^{(k)} \langle -|_{CD}^{(k)}$  is the initial state of the qubit system. Firstly suppose the model Hamiltonian of entanglement transfer from CV system to qubit system or vice versa is

$$H_1 = \hbar\Omega (\sqrt{n}a^+ \sigma_- + a\sqrt{n}\sigma_+), \quad (1)$$

where  $a$  and  $a^+$  are the photon annihilation and generation operators respectively,  $n = a^+a$ ,  $\sigma_-$  and  $\sigma_+$  are operators which convert the atom (the qubit) from its excited state  $|+\rangle$  to ground state  $|-\rangle$  and from ground state to excited state respectively. The Hamiltonian (1) can be considered as a kind of nonlinear Jaynes-Cummings model[2] [3]. Then  $\exp[-\frac{i}{\hbar}H_1 t_1] |m, -\rangle = \cos(m\Omega t_1) |m, -\rangle - i \sin(m\Omega t_1) |m-1, +\rangle$ . If the interaction time  $t_1$  is adjusted in such a way that  $\Omega t_1 = \pi/2$  then  $\exp[-\frac{i}{\hbar}H_1 t_1] |2m, -\rangle = (-1)^m |2m, -\rangle$  and  $\exp[-\frac{i}{\hbar}H_1 t_1] |2m+1, -\rangle = -i(-1)^m |2m, +\rangle$ . Apply the evolution operator  $U_1(t_1) = \exp[-\frac{i}{\hbar}(H_{1AC}+H_{1BD})t_1]$

to the state  $|\Psi\rangle_{AB} |--\rangle_{CD}^{(1)}$ , we have

$$U_1(t_1) |\Psi\rangle_{AB} |--\rangle_{CD}^{(1)} = |\Psi\rangle_{AB}^{(1)} |\Phi\rangle_{CD}^{(1)} \quad (2)$$

with  $|\Psi\rangle_{AB}^{(1)} = \sqrt{1-\lambda^4} \sum_{m=0}^{\infty} \lambda^{2m} |2m\rangle_A |2m\rangle_B$  and  $|\Phi\rangle_{CD}^{(1)} = \frac{1}{\sqrt{1+\lambda^2}} (|--\rangle_{CD}^{(1)} - \lambda |++\rangle_{CD}^{(1)})$ . It should be noticed that the state after evolution is a product state of CV system state and two qubit state. The CV state  $|\Psi\rangle_{AB}^{(1)}$  has even number of photons in each mode. We can separate the two qubit state  $|\Phi\rangle_{CD}^{(1)}$  from the combined state, then append another vacuum two qubit state  $|--\rangle_{CD}^{(2)}$  of  $CD$  partite to state  $|\Psi\rangle_{AB}^{(1)}$ , the new state will be  $|\Psi\rangle_{AB}^{(1)} |--\rangle_{CD}^{(2)}$ . We would design another interaction Hamiltonian to assign the entanglement of CV state to two qubit state. The Hamiltonian will be  $H_2 = \hbar\Omega \left( \sqrt{n}a^+ \frac{1}{\sqrt{n}}a^+ \sigma_- + a \frac{1}{\sqrt{n}}a \sqrt{n}\sigma_+ \right)$ , the evolution will be  $U_2(t_2) |\Psi\rangle_{AB}^{(1)} |--\rangle_{CD}^{(2)} = |\Psi\rangle_{AB}^{(2)} |\Phi\rangle_{CD}^{(2)}$  with the interaction time  $t_2 = \pi/(4\Omega)$ , and  $|\Psi\rangle_{AB}^{(2)} = \sqrt{1-\lambda^8} \sum_{m=0}^{\infty} \lambda^{4m} |4m\rangle_A |4m\rangle_B$ ,  $|\Phi\rangle_{CD}^{(2)} = \frac{1}{\sqrt{1+\lambda^4}} (|--\rangle_{CD}^{(2)} - \lambda^2 |++\rangle_{CD}^{(2)})$ . Then we move from the second two qubit to the vacuum state of the third two qubit of  $CD$  partite and so on. The  $k$ -th Hamiltonian will be  $H_k = \hbar\Omega [n(\frac{1}{\sqrt{n}}a^+)^{2^{k-1}} \sigma_- + (a \frac{1}{\sqrt{n}})^{2^{k-1}} n\sigma_+]$  and interaction time  $t_k = \pi/(2^k\Omega)$ . The whole state reads

$$U_k(t_k) \cdots U_2(t_2) U_1(t_1) |\Psi\rangle_{AB} (|--\rangle^{(1)} |--\rangle^{(2)} \cdots |--\rangle^{(k)})_{CD} = |\Psi\rangle_{AB}^{(k)} (|\Phi\rangle^{(1)} |\Phi\rangle^{(2)} \cdots |\Phi\rangle^{(k)})_{CD}, \quad (3)$$

with  $|\Psi\rangle_{AB}^{(k)} = \sqrt{1-\lambda^{2^{k+1}}} \sum_{m=0}^{\infty} \lambda^{2^k m} |2^k m\rangle_A |2^k m\rangle_B$ ,  $|\Phi\rangle_{CD}^{(k)} = \frac{1}{\sqrt{1+\lambda^{2^k}}} (|--\rangle_{CD}^{(k)} - \lambda^{2^{k-1}} |++\rangle_{CD}^{(k)})$ . The entanglement transferred to qubits system is

$$\begin{aligned} E\left(\prod_{j=1}^K |\Phi\rangle_{CD}^{(j)}\right) &= \sum_{j=1}^K E(|\Phi\rangle_{CD}^{(j)}) = \sum_{j=1}^K [\log(1+\lambda^{2^j}) - \frac{\lambda^{2^j}}{1+\lambda^{2^j}} 2^j \log \lambda] \\ &= \log \frac{1-\lambda^{2^{K+1}}}{1-\lambda^2} - \left(\frac{\lambda^2}{1-\lambda^2} - \frac{2^K \lambda^{2^{K+1}}}{1-\lambda^{2^{K+1}}}\right) \log \lambda^2. \end{aligned} \quad (4)$$

The entanglement remained at the CV system is  $E(|\Psi\rangle_{AB}^{(K)}) = -\log(1-\lambda^{2^{K+1}}) - \frac{2^{K+1} \lambda^{2^{K+1}}}{1-\lambda^{2^{K+1}}} \log \lambda$ . The total entanglement remains unchanged for each  $K$ ,  $E\left(\prod_{j=1}^K |\Phi\rangle_{CD}^{(j)}\right) + E(|\Psi\rangle_{AB}^{(K)}) = E(|\Psi\rangle_{AB})$ . When  $K \rightarrow \infty$ ,  $\lambda^{2^{K+1}} \rightarrow 0$ , thus  $E(|\Psi\rangle_{AB}^{(K)}) \rightarrow 0$ , the entanglement transferred to the qubit system tends to  $E(|\Psi\rangle_{AB})$ . The entanglement is perfectly transferred. The entanglement transfer is depicted in Fig.1 for different value of receiving qubit pair number  $K$ .

### 3 Reverse conversion of entanglement

In the reverse conversion, we have the initial state  $|\phi_1\rangle_{CD} |\phi_2\rangle_{CD} \cdots |\phi_K\rangle_{CD}$ , where  $|\phi_i\rangle_{CD} = a_{00}^i |--\rangle^{(i)} + a_{01}^i |-\rangle^{(i)} + a_{10}^i |+\rangle^{(i)} + a_{11}^i |+\rangle^{(i)}$ . The process of entanglement transfer is to transfer firstly the higher qubit pair ( $K$ -th) to the CV bipartite state then the lower. The result of conversion will be  $U_1^+(t_1) |\phi_1\rangle_{CD} U_2^+(t_2) |\phi_2\rangle_{CD} \cdots U_K^+(t_K) |\phi_K\rangle_{CD} |00\rangle_{AB} = |\psi\rangle_{AB} \prod_{k=1}^K |--\rangle_{CD}^{(k)}$ , where  $|\psi\rangle_{AB} = \sum_{n_1, \dots, n_K, m_1, \dots, m_K=0}^1 \prod_{j=1}^K (-1)^{m_{j+1}+n_{j+1}} i^{m_j+n_j} a_{n_j m_j}^j |n_K \cdots n_1, m_K \cdots m_1\rangle$ , with  $m_j, n_j = 0, 1$  and  $m_{K+1} = n_{K+1} = 0$ . We denoted  $n = \sum_{j=1}^K n_j 2^{j-1}$  for later use. The Entanglement of the state  $|\psi\rangle_{AB}$  is equal to that of a state  $|\psi'\rangle = \sum_{n_1, m_1=0}^1 i^{m_1+n_1} a_{n_1 m_1}^1 |n_1, m_1\rangle \prod_{j=2}^K (\sum_{n_j, m_j=0}^1 (-i)^{m_j+n_j} a_{n_j m_j}^j |n_j, m_j\rangle)$ , thus it is equal to the sum of entanglements of qubit pairs. We have  $E(|\psi\rangle_{AB}) = \sum_{j=1}^K E(|\phi_j\rangle_{CD})$ .

The conversion procedure will convert a general qubit pair product state  $\rho_{CD}^{(1)} \otimes \rho_{CD}^{(2)} \otimes \cdots \otimes \rho_{CD}^{(K)}$  into a continuous variable state  $\rho_{AB}$  while keeping the entanglement due to local unitary operations. The process of reverse entanglement conversion is to convert firstly the highest two qubits ( $K$ -th) to the CV system then the lower. The combined state will evolve to

$$U_1^\dagger(t_1) \{ \{ U_2^\dagger(t_2) \cdots \{ U_K^\dagger(t_K) |00\rangle \langle 00| \otimes \rho_{CD}^{(K)} U_K(t_K) \} \cdots \otimes \rho_{CD}^{(2)} U_2(t_2) \otimes \rho_{CD}^{(1)} \} U_1(t_1), \quad (5)$$

where  $|00\rangle\langle 00|$  is the initial state of the field ( $A$  and  $B$  bipartite). Since

$$\begin{aligned}\exp\left(\frac{i}{\hbar}H_K t_K\right)|0,-\rangle_K &= |0,-\rangle_K \\ \exp\left(\frac{i}{\hbar}H_K t_K\right)|0,+\rangle_K &= \cos(2^{K-1}\Omega t_K)|0,+\rangle_K + i \sin(2^{K-1}\Omega t_K)|2^{K-1},-\rangle_K.\end{aligned}\quad (6)$$

The evolution time is so chosen that  $\cos(2^{K-1}\Omega t_K) = 0$ , we choose  $2^{K-1}\Omega t_K = \pi/2$  as before. Then  $\exp\left(\frac{i}{\hbar}H_K t_K\right)|0,+\rangle_K = i|2^{K-1},-\rangle_K$ . The first step evolution will be  $U_K^\dagger(t_K)|00\rangle\langle 00| \otimes \rho_{CD}^{(K)} U_K(t_K) = \rho_{AB}^{(K)} \otimes |-\rangle_{CD,CD}^{(K)}\langle--|$ , where  $U_K(t_K) = \exp[-\frac{i}{\hbar}(H_{KAC}+H_{KBD})t_K]$  as before. The basis of  $\rho_{AB}^{(K)}$  are  $|n_K 2^{K-1}, m_K 2^{K-1}\rangle$ , with  $n_K, m_K = 0$  or  $1$ . We see that all the information of qubit state  $\rho_{CD}^{(K-1)}$  is transferred to the field, leave the two qubit state a definite blank state. Moreover, the combined state is a direct product of the field and two qubit state, thus the  $K-th$  qubit pair can be dropped after the evolution. The next step is to transfer  $\rho_{CD}^{(K-1)}$  to the remained field  $\rho_{AB}^{(K)}$ . Since when  $2^{K-2}\Omega t_{K-1} = \pi/2$ , we have The state after the evolution will be  $\rho_{AB}^{(K-1)} \otimes |-\rangle_{CD,CD}^{(K-1)}\langle--|\otimes|-\rangle_{CD,CD}^{(K)}\langle--|$ . The basis of  $\rho_{AB}^{(K)}$  are  $|n_K 2^{K-1} + n_{K-1} 2^{K-2}, m_K 2^{K-1} + m_{K-1} 2^{K-2}\rangle$ , with  $n_{K-1}, m_{K-1} = 0$  or  $1$ . The quantum state of a pair of the two qubits  $\rho_{CD}^{(K)} \rho_{CD}^{(K-1)}$  are transferred to  $\rho_{AB}^{(K-1)}$ . When all the two qubits are transferred to the field, we get at last a bipartite quantum CV state  $\rho = \rho_{AB}^{(1)}$  while leaving all the two qubit series in the lower energy level state  $\prod_{k=1}^K |-\rangle_{CD}^{(k)}$ . Thus the reverse conversion procedure will convert a general qubit pair product state  $\rho_{CD}^{(1)} \otimes \rho_{CD}^{(2)} \otimes \dots \otimes \rho_{CD}^{(K)}$  into a continuous variable bipartite state  $\rho$  while keeping the entropy of the whole state. This is due to the fact that local unitary transformation does not change the entanglement. We here divide the system into  $AC$  and  $BD$  subsystems. The reverse conversion is perfect.

## 4 Entanglement conversion of continuous variable Werner state

CV Werner state is a mixture of the two mode squeezed vacuum state and the two mode thermal state [2]

$$\rho_W = p\rho_{TMSV} + (1-p)\rho_T, \quad 0 \leq p \leq 1.$$

Where

$$\begin{aligned}\rho_{TMSV} &= (1-\lambda^2) \sum_{m,n=0}^{\infty} \lambda^{m+n} |mm\rangle\langle nn|, \\ \rho_T &= (1-v)^2 \sum_{m,n=0}^{\infty} v^{m+n} |mn\rangle\langle mn|.\end{aligned}$$

The sufficient conditions of inseparability and separability as well as other physical properties have been displayed. Denote partial transposed state of  $\rho_W$  as  $\rho_W^{T_A}$ . The eigenvalues of  $\rho_W^{T_A}$  are [2]

$$\begin{aligned}x^{(l)} &= p(1-\lambda^2)\lambda^{2l} + (1-p)(1-v)^2v^{2l}, \quad l = 0, 1, \dots \\ x_{1,2}^{(mn)} &= (1-p)(1-v)^2v^{m+n} \pm p(1-\lambda^2)\lambda^{m+n}, \quad m \neq n; m, n = 0, 1, \dots.\end{aligned}$$

The logarithmic negativity (LN) as an entanglement measure of the state  $\rho_W$  can be calculated [4]. The entanglement measured by LN is  $E_{LN} = \log_2 \left\| \rho_W^{T_A} \right\|_1$ , where  $\|A\|_1 = \text{Tr}\sqrt{A^\dagger A}$ . In the simplest case of  $v = \lambda$  (although LN can be worked out for any values of the parameters), the inseparable condition is  $p > (1-\lambda)/2$ , we have

$$E_{LN} = \log_2 \left[ p \frac{1+\lambda}{1-\lambda} + (1-p) \frac{1-\lambda}{1+\lambda} \right].$$

The conversion of the CV Werner state to a serial of two qubits system leads to the result state

$$\rho_{WCD} = p \bigotimes_{k=1}^K (|\Phi\rangle_{CD,CD}^{(k)} \langle \Phi|^{(k)}) + (1-p) \bigotimes_{k=1}^K (\varrho_C^{(k)} \otimes \varrho_D^{(k)}).$$

Where  $\varrho_C^{(k)} = \frac{1}{1+v^{2k-1}} (|-\rangle_{C,C}^{(k)} \langle -| + v^{2k-1} |+\rangle_{C,C}^{(k)} \langle +|)$  and  $\varrho_D^{(k)}$  is the similar. For convenience, we denote  $|-\rangle$  and  $|+\rangle$  as  $|0\rangle$  and  $|1\rangle$ , then the basis of  $C$  system (as well as  $D$  system) can be denoted as  $|n_1 \dots n_K\rangle_C$  with  $n_k = 0, 1$ . The basis can be further simplified as  $|n\rangle_C$  with  $n = \sum_{k=1}^K n_k 2^{k-1}$ . Thus

$$\rho_{WCD} = p \frac{1-\lambda^2}{1-\lambda^{2^{K+1}}} \sum_{n,m=0}^{2^K-1} (-1)^{\sum_{k=1}^K (n_k+m_k)} \lambda^{m+n} |mm\rangle \langle nn| + (1-p) \frac{(1-v)^2}{(1-v^{2^K})^2} \sum_{n,m=0}^{2^K-1} v^{m+n} |mn\rangle \langle mn|.$$

The partially transposed matrix  $\rho_{WCD}^{T_C}$  has a block diagonal form with  $1 \times 1$  blocks in one-dimensional subspaces spanned by vectors  $\{|mm\rangle\}$ ,  $m = 0, 1, \dots$  and  $2 \times 2$  blocks in two-dimensional subspaces spanned by vectors  $\{|mn\rangle, |nm\rangle, m \neq n\}$ ,  $m, n = 0, 1, \dots$ . Consequently, the eigenvalues of the partially transposed matrix  $\rho_{WCD}^{T_C}$  can easily be calculated as roots of quadratic equations and read

$$\begin{aligned} x^{(l)} &= p \frac{1-\lambda^2}{1-\lambda^{2^{K+1}}} \lambda^{2l} + (1-p) \frac{(1-v)^2}{(1-v^{2^K})^2} v^{2l}, \quad l = 0, 1, \dots \\ x_{1,2}^{(mn)} &= (1-p) \frac{(1-v)^2}{(1-v^{2^K})^2} v^{m+n} \pm p \frac{1-\lambda^2}{1-\lambda^{2^{K+1}}} \lambda^{m+n}, \quad m \neq n; \quad m, n = 0, 1, \dots . \end{aligned}$$

After the conversion, the entanglement measured by LN for the case of  $v = \lambda$  will be

$$E_{LNCD} = \log_2 [p \frac{1+\lambda}{1-\lambda} \frac{1-\lambda^{2^K}}{1+\lambda^{2^K}} + (1-p) \frac{1-\lambda}{1+\lambda} \frac{1+\lambda^{2^K}}{1-\lambda^{2^K}}].$$

The entanglement transfer is shown in Fig.2 for different value of receiving qubit pair number  $K$ .

## 5 Conclusion

The entanglement of continuous variable system can be converted to the entanglement of a serial of qubit pairs. The conversion error can be made arbitrary low if the series is long enough. The reverse conversion is perfect as far as the different qubit pairs are not correlated before the reverse conversion. For a typical two mode squeezed vacuum state prepared in laboratory (the squeezing parameter  $r < 3$ ), the qubit system with eight pairs of qubits is sufficiently good in simulation of the original CV system. This is indicated by our calculations of the entanglement transfer of two mode squeezed vacuum state or CV Werner state. In quantum information theory, the entanglement of two qubits changes from 0 to 1, this flexibility makes the error of conversion quite small as we can see from the figures.

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## References

- [1] Paternostro M, Falci G, Kim M and Palma G M 2004 Phys. Rev. B **69** 214502
- [2] Mista Jr. L, Filip R and Fiurasek J 2002 Phys. Rev. A **65** 062315
- [3] Chen X Y, Han L and Jiang L Z 2006 Phys. Lett. A **359** 587
- [4] Vidal G and Werner R F 2002 Phys. Rev. A **65** 032314

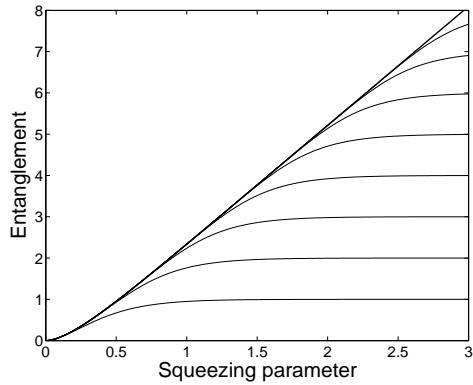


Figure 1: The thick line is for CV state, The thin lines from bottom to top are for the entanglement transferred of  $K=1,2,\dots,8$  respectively.

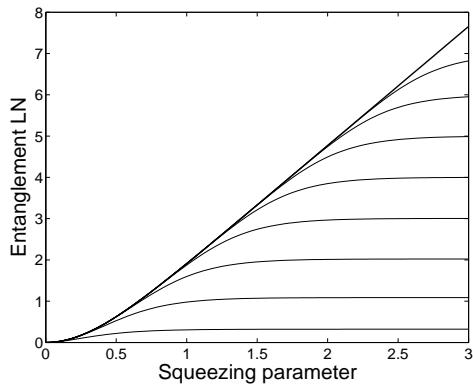


Figure 2: The thick line is for CV state, The thin lines from bottom to top are for the entanglement transferred of  $K=1,2,\dots,8$  respectively.  $p=0.5$ .